PMM U.S.S.R., Vol.47, No.6, pp. 756-760, 1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 532.29

DIFFRACTION OF KELVIN WAVES IN A CHANNEL WITH A SEMI-INFINITE WALL*

V.I. PLIS

The Wiener-Hopf method is used to derive an exact solution of the problem of diffraction of Kelvin waves in a rotating channel containing a semi-infinite wall. A numerical analysis of the solution is carried out. The nature of the waves propagating in the channel is considered.

1. Statement of the problem. Let a channel $-b<y<a, -\infty<x<+\infty$ situated on a plane Earth, rotating anticlockwise at an angular velocity ω, have a semi-infinite wall $y=0, x<0$ (Fig.1). The channel depth is constant and equal to h. The axis of rotation is normal to the xy plane and passes through the point with coordinates (0, 0).

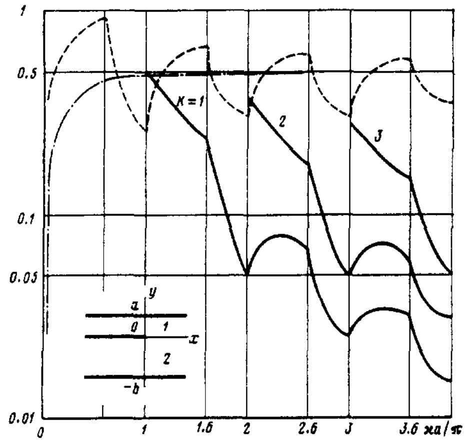

Fig.1

We will consider in this channel the steady-state wave motions of the liquid surface i.e. we will assume that the elevations $\xi(x,y,t)$ depend harmonically on time $\xi(x,y)\exp(i\sigma t)$. Let us consider the case when $\sigma>2\omega$. In the linear theory of long surface waves /1/ the function $\xi(x,y)$ is the solution of the wave equation

$$(\Delta+\varkappa^2)\,\xi(x,y)=0,\ \varkappa^2=(\sigma^2-4\omega^2)/(gh)$$

where g is the acceleration of free fall and Δ is the two-dimensional Laplace operator.

Suppose that along the semi-infinite channel wall in the region $0<y<a,\ -\infty<x<0$ a Kelvin wave propagates with unit amplitude

$$\xi_0(x,y)=\exp(i\eta\varkappa x-l\eta\varkappa y);\quad l=\frac{2\omega}{\sigma}<1,\quad \eta=(1-l^2)^{-1/2} \tag{1.1}$$

We will investigate the wave motions in the channel generated by the diffraction of that wave at the edge of the semi-infinite wall.

We divide the channel into two regions, as shown in Fig.1. In region 1 $(0<y<a,\ -\infty<x<+\infty)$ we represent the total elevation amplitude in the form $\xi_0+\xi_1$, where ξ_0 is the incident wave and ξ_1 the diffracted wave. In region 2 $(-b<y<0,\ -\infty<x<+\infty)$ we will represent the unknown elevation amplitude by ξ_2. For the unknown functions ξ_j $(j=1,2)$ we obtain the problem of determining the solutions of equations

$$(\Delta+\varkappa^2)\,\xi_j(x,y)=0\quad (j=1,2) \tag{1.2}$$

that satisfy the boundary conditions on the channel walls and the conditions of continuity of the y components of the velocities and elevations on the continuation of the infinite wall

$$v_1(x,a-0)=0,\ -\infty<x<+\infty;\ v_1(x,+0)=0,\ -\infty<x<0 \tag{1.3}$$

$$v_2(x,-b+0)=0,\ -\infty<x<+\infty;\ v_2(x,-0)=0,\ -\infty<x<0$$

$$v_1(x,+0)=v_2(x,-0),\ 0<x<+\infty \tag{1.4}$$

$$\xi_0(x,+0)+\xi_1(x,+0)=\xi_2(x,-0),\ 0<x<+\infty$$

Here $v_j(x,y)$ is the velocity component of the liquid parallel to the y axis and connected with $\xi_j(x,y)$ by the relation

$$v_j(x,y)=-\frac{\sigma}{\varkappa^2h}\left(l\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\xi_j(x,y) \tag{1.5}$$

Finally, the diffracted waves must satisfy the condition on the edge /2/

$$\xi_j\sim r^{1/2},\ r\to 0,\ r=(x^2+y^2)^{1/2} \tag{1.6}$$

and the radiation condition: the solution at infinity must contain only diverging waves.

It can be shown that in the class of bounded functions problem (1.1)-(1.6) has a unique solution.

---

*Prikl.Matem.Mekhan., 47, 6, 947-953, 1983

2. The system of paired integral equations and its solution. Equations (1.1) – (1.6) will be solved by the Wiener-Hopf method /3/. For this we assume that the wave number $\varkappa$ has a small positive imaginary part, i.e. $\varkappa = \varkappa_0 + i\varepsilon$, and in the final results we let ε tend to zero. The introduction into $\varkappa$ of an imaginary part corresponds to the assumption of energy dissipation in the liquid.

We introduce the unknown functions $A(\alpha)$, $B(\alpha)$, $G(\alpha)$, $D(\alpha)$, $Z_1(\alpha)$, $Z_2(\alpha)$, $Z_2'(\alpha)$, $Z_3(\alpha)$ of the complex variable α , using the formulas

$$\begin{aligned}
\xi_1(x, y) &= \int_{-\infty}^{+\infty} \exp(i\alpha x)[A(\alpha)\sin\gamma(y-a) + B(\alpha)\sin\gamma y]\,d\alpha \\
\xi_2(x, y) &= \int_{-\infty}^{+\infty} \exp(i\alpha x)[G(\alpha)\sin\gamma(y+b) + D(\alpha)\sin\gamma y]\,d\alpha \\
\xi_1(x, a-0) &= \int_{-\infty}^{+\infty} \exp(i\alpha x) Z_1(\alpha)\,d\alpha \\
\xi_1(x, +0) &= \int_{-\infty}^{+\infty} \exp(i\alpha x) Z_2'(\alpha)\,d\alpha \\
\xi_2(x, -0) &= \int_{-\infty}^{+\infty} \exp(i\alpha x) Z_2(\alpha)\,d\alpha \\
\xi_2(x, -b+0) &= \int_{-\infty}^{+\infty} \exp(i\alpha x) Z_3(\alpha)\,d\alpha
\end{aligned} \tag{2.1}$$

where $\gamma = (\varkappa^2 - \alpha^2)^{1/2}$ and the branch of the root is selected so that $\operatorname{Im}\gamma > 0$.

It can be seen that these functions are not independent

$$\begin{aligned}
A(\alpha) = -Z_2'(\alpha)/\sin\gamma a, \quad B(\alpha) = Z_1(\alpha)/\sin\gamma a \\
G(\alpha) = Z_2(\alpha)/\sin\gamma b, \quad D(\alpha) = -Z_3(\alpha)/\sin\gamma b
\end{aligned} \tag{2.2}$$

From the conditions for the wall to be impermeable $y = a$, $y = -b$ we have

$$Z_1(\alpha) = \frac{\gamma Z_2'(\alpha)}{\gamma\cos\gamma a + \alpha l\sin\gamma a}, \quad Z_3(\alpha) = \frac{\gamma Z_2(\alpha)}{\gamma\cos\gamma b - \alpha b\sin\gamma b} \tag{2.3}$$

We will introduce the new unknown function $V(\alpha)$ using the formula

$$v_1(x, 0) = -\frac{\sigma}{\varkappa^2 h}\int_{-\infty}^{+\infty} \exp(i\alpha x) V(\alpha)\,d\alpha \tag{2.4}$$

Applying (1.5) to the integral representations for elevations (2.1) we obtain, taking (2.2)–(2.4) into account, the dependence of $Z_2'(\alpha)$ and $Z_2(\alpha)$ on $V(\alpha)$

$$\begin{aligned}
Z_2'(\alpha) &= -i\frac{\gamma\cos\gamma a + \alpha l\sin\gamma a}{\sin\gamma a\,(\gamma^2 + \alpha^2 l^2)} V(\alpha) \\
Z_2(\alpha) &= i\frac{\gamma\cos\gamma b - \alpha l\sin\gamma b}{\sin\gamma b\,(\gamma^2 + \alpha^2 l^2)} V(\alpha)
\end{aligned}$$

Substituting the integral formulas into the second boundary condition (1.4) and using the condition of impermeability of the semi-infinite wall, we obtain the system of paired integral equations

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{\exp(i\alpha x) L(\alpha)}{\alpha^2 - \eta^2\varkappa^2} V(\alpha)\,d\alpha = \frac{ab}{a+b}\frac{1}{\eta^2}\exp(i\eta\varkappa x), \quad x > 0 \\
&\int_{-\infty}^{+\infty} \exp(i\alpha x) V(\alpha)\,d\alpha = 0, \quad x < 0 \\
&L(\alpha) = \frac{\sin\gamma(a+b)}{\gamma(a+b)}\,\frac{\gamma a}{\sin\gamma b}\,\frac{\gamma b}{\sin\gamma b}
\end{aligned} \tag{2.5}$$

To solve (2.5) we factorize the kernel of the integral equation $L(\alpha)$, i.e. we represent it in the form $L(\alpha) = L_+(\alpha)L_-(\alpha)$, where the function $L_+(\alpha)$ is analytic and has no zeros in the upper half-plane of the complex variable α , and the function $L_-(\alpha)$ has the same properties in the lower half-plane of the complex variable α . Factorization of the function $\sin\gamma a/\gamma a$ has often been used in the literature /3, 4/, hence we present here the final result of this procedure for the kernel of $L(\alpha)$

$$L_+(\alpha) = \left(\frac{\varkappa ab}{c}\,\frac{\sin\varkappa c}{\sin\varkappa a\sin\varkappa b}\right)^{1/2}\exp(P(\alpha)) \times \tag{2.6}$$

$$\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\alpha_{nc}}\right)\exp\left(\frac{i\alpha c}{\pi n}\right)\left[\left(1+\frac{\alpha}{\alpha_{na}}\right)\exp\left(\frac{i\alpha a}{\pi n}\right)\left(1+\frac{\alpha}{\alpha_{nb}}\right)\times\exp\left(\frac{i\alpha b}{\pi n}\right)\right]^{-1}$$

$$c=a+b,\quad P(\alpha)=\frac{i\alpha}{\pi}(c\ln c-a\ln a-b\ln b)$$

$$\alpha_{nd}=\left[\varkappa^2-\left(\frac{\pi n}{d}\right)^2\right]^{1/2}\quad (d=a,b,c)$$

We will seek a solution of (2.5) in the form

$$V(\alpha)=Q/L_-(\alpha) \tag{2.7}$$

where Q is an unknown constant. With this selection of the function $V(\alpha)$ the second of the equations of (2.5) is identically satisfied. To determine the constant Q we substitute (2.7) into the first of the integral equations of (2.5) and calculate the residue in the strip $\alpha=\eta\varkappa$. We obtain

$$Q=\frac{\varkappa ab}{\pi\eta(a+b)L_+(\eta\varkappa)} \tag{2.8}$$

The solution (2.7), (2.8) satisfies the condition on the edge (1.6), which by the theorem on the relation between the asymptotic form of the function and its Fourier image /5/, takes for the function $V(\alpha)$ the following form:

$$V(\alpha)\sim\alpha^{-1/2}\text{ as }\alpha\to\infty$$

Knowing the explicit expression for $V(\alpha)$ it is possible to construct the formulas for the elevations of the surface of the liquid in the channel.

3. Formulas for the elevations. We will start by investigating the elevations in the branched part of the channel, i.e. when $x<0$. Starting with (2.1) we can obtain the following integral representation for the elevations in region *1*:

$$\xi_1(x,y)=I(a) \tag{3.1}$$

$$I(d)=i\eta^2\int_{-\infty}^{+\infty}\frac{\exp(i\alpha x)\left[\gamma\cos\gamma(y-d)-\alpha l\sin\gamma(y-d)\right]}{\sin\gamma d\,(\alpha^2-\eta^2\varkappa^2)}V(\alpha)\,d\alpha$$

To evaluate the integral in (3.1) when $x<0$ it is sufficient to determine the residues of the integrand of the function at simple poles $-\eta\varkappa,\ -\alpha_{na}$ $(n=1,2,\ldots)$. As the result, we obtain

$$\xi_1(x,y)=\pi\eta^2 l\frac{V(-\eta\varkappa)}{\operatorname{sh}(l\eta\varkappa a)}\exp\left[-i\eta\varkappa x+l\eta\varkappa(y-a)\right]-\Sigma_a \tag{3.2}$$

$$\Sigma_a=2\pi\sum_{n=1}^{\infty}\frac{\gamma_{na}V(-\alpha_{na})}{\alpha_{na}a(\gamma_{na}^2+\alpha_{na}^2l^2)^{1/2}}\sin(\gamma_{na}y+\varphi_{na})\exp(-i\alpha_{na}x)$$

$$\sin\varphi_{na}=\frac{\gamma_{na}}{(\gamma_{na}^2+\alpha_{na}^2l^2)^{1/2}},\quad\cos\varphi_{na}=\frac{\alpha_{na}l}{(\gamma_{na}^2+\alpha_{na}^2l^2)^{1/2}},\quad\gamma_{na}=\frac{\pi n}{a}$$

In region *2* the elevations are defined by the integral relation

$$\xi_2(x,y)=I(-b) \tag{3.3}$$

When $\operatorname{Im}\alpha<0$, the integrand has simple poles at the points $-\eta\varkappa,\ -\alpha_{nb}$ $(n=1,2,\ldots)$. In the expression for ξ_2 waves propagating in region *2* in the negative direction of the x axis correspond to them

$$\xi_2(x,y)=\frac{2l\eta\varkappa ab\exp(-i\eta\varkappa x+l\eta\varkappa y)}{(a+b)\left[1-\exp(2l\eta\varkappa b)\right]L_+^2(\eta\varkappa)}-\Sigma_b \tag{3.4}$$

where the expression for Σ_b is similar to Σ_a (with a replaced by b).

In the half-plane $\operatorname{Im}\alpha>0$ the integrand in (3.3) has simple poles at the points $\eta\varkappa$, α_{nc} $(n=1,2,\ldots)$. The respective wave motions in region *2* when $x>0$ are defined as follows:

$$\xi_2(x,y)=\frac{\operatorname{sh}(l\eta\varkappa a)}{\operatorname{sh}(l\eta\varkappa c)}\exp\left[i\eta\varkappa x-l\eta\varkappa(y+b)\right]+ \tag{3.5}$$

$$2\pi\sum_{n=1}^{\infty}\frac{\operatorname{res}V(\alpha_{nc})}{\sin\gamma_{nc}b\,(\gamma_{nc}^2+\alpha_{nc}^2l^2)^{1/2}}\sin\left[\gamma_{nc}(y+b)-\varphi_{nc}\right]\exp(i\alpha_{nc}x)$$

where the expressions for φ_{nc}, γ_{nc} are similar to those in (3.2) (with a replaced by c); and $\operatorname{res}V(\alpha_{nc})$ is the residue of the function $V(\alpha)$ when $\alpha=\alpha_{nc}$.

It can be shown that (3.5) also holds for the elevations in region *1* when $x>0$, taking the incident Kelvin wave into account.

We will now investigate the nature of the surface wave motions in the channel. In each of formulas (3.2), (3.4), and (3.5) the first term defines a Kelvin wave and the infinite sum corresponds to progressive and decaying waves. Progressive waves correspond to real values of α_{nd} $(d = a, b, c)$ and, exponentially decaying waves correspond to imaginary values. For a given value of the dimensionless channel width $\varkappa d$ the number of progressing waves is equal to the integral part of $\varkappa d/\pi$.

Thus, depending on the relation between the quantities $a, b, c, \varkappa, \pi$ one or another number of undamped waves can propagate in the channel, in which case Kelvin waves will exist for any $\varkappa d$.

4. Propagation of Kelvin waves along an infinite wall.

Suppose that in the channel shown in Fig.1 along the wall $y = -b$ a Kelvin wave of unit amplitude

$$\xi_0(x, y) = \exp\,[i\eta\varkappa x - l\eta\varkappa\,(y + b)]$$

propagates in the positive direction of the x axis.

Solution of the problem of diffraction of this wave using the method described in Sects. 1 and 2 leads to a system of paired integral equations

$$\int\limits_{-\infty}^{+\infty} \frac{\exp(i\alpha x)\,L(\alpha)}{\alpha^2 - \eta^2\varkappa^2} V(\alpha)\,d\alpha = -\frac{i}{\eta^2}\frac{ab}{a+b}\exp(i\eta\varkappa x - l\eta\varkappa b), \quad x > 0 \tag{4.1}$$

$$\int\limits_{-\infty}^{+\infty} \exp(i\alpha x)\,V(\alpha)\,d\alpha = 0, \quad x < 0$$

In these equations the notation is exactly the same as in Sects.1 and 2. Equation (4.1) is solved by the method of factorization, and the unknown function is sought in the form $V(\alpha) = R/L_{-}(\alpha)$. It is obvious that $R = -i\exp(-l\eta\varkappa b)Q$. The wave motions that occurs in this case are the same as those described in Sect.3, except that their amplitude is $\exp(l\eta\varkappa b)$ times less, i.e. the right-hand sides of (3.2), (3.4), and (3.5) are multiplied by $-i\exp(-l\eta\varkappa b$

The problem of the diffraction of a Kelvin wave propagating along an infinite wall in a branched channel was, first, solved by the Wiener-Hopf method in the Jones interpretation in /6/, and somewhat later in /7/. The equations obtained there for the wave amplitude in a channel are the same as those obtained by multiplying the amplitudes in (3.2), (3.4), and (3.5) by $-i\exp(-l\eta\varkappa b)$.

It should be noted that in /7/, as well as in /8/ on a similar theme by the same author, it was stated that the amplitude of the n-th progressive wave in the channel approaches infinity as $\alpha_{nd} \to 0$, i.e. when $\varkappa d \to \pi n$ $(d = a, b, c)$, and the formulas for the elevations hold for $\varkappa d$ not too close to πn, because in the expressions for the amplitudes of the progressive waves (3.2), (3.4) and (3.5) the wave parameters α_{nd} are in the denominator. However, the numerator in these expressions always contains a multiplier of the form $(\sin\varkappa d)^{1/2}$ that approaches zero as $\varkappa d \to \pi n$ at the same rate as α_{nd}. Hence, the amplitude of any progressing wave in the channel is always finite, and the solution (3.2), (3.4), and (3.5) holds for any $\varkappa d$.

5. Interpretation of the results of a numerical analysis.

The amplitudes of the waves in the channel wave investigated numerically. For this the infinite products in (2.6) were replaced by finite products with N multipliers.

We will estimate the error of such reduction. We denote by P_N the finite product, typical for problems of wave diffraction in channels /6—11/,

$$P_N = \prod_{n=1}^{N}\left(1 + \frac{\alpha}{\alpha_{nd}}\right)\exp\left(\frac{i\alpha d}{\pi n}\right) \tag{5.1}$$

Then the relative error of the reduction is

$$\delta_N = \left|\frac{P_\infty - P_N}{P_\infty}\right| = \left|1 - \prod_{n=N+1}^{\infty}\left(1 + \frac{\alpha}{\alpha_{nd}}\right)^{-1}\exp\left(-\frac{i\alpha d}{\pi n}\right)\right|$$

In the product (5.1) we have $\alpha_{nd} \simeq i\pi n/d$ for large n. If we neglect terms whose order of magnitude are less than $\alpha d/\pi n$, we obtain the following estimate for the relative reduction error:

$$\varepsilon_N \simeq \left|\left\{\prod_{n=N+1}^{\infty}\left[1 - \left(\frac{\alpha d}{\pi n}\right)^2\right]\right\} - 1\right| = \left|\frac{\alpha d}{\pi}\right|^2 \sum_{n=N+1}^{\infty}\frac{1}{n^2} < \frac{|\alpha|^2 d^2}{\pi^2 N}$$

For values $\alpha d/\pi = 1$ the quantity ε_N will be less than 1% when $N = 100$.

In Fig.1 we show the dependence on $\varkappa a$ for $\varkappa b/\pi = 0.4$ of the amplitude of the Kelvin wave propagating in the non-branching part of the channel (the dash-dot line), the dependence on $\varkappa a$ of the amplitude of the Kelvin wave propagating in region *2* in the negative direction of the x axis (the dashed line), and the dependence on $\varkappa a$ of the amplitude of progressive waves in region l when $x < 0$ (the solid line). It can be seen that the amplitude of the Kelvin wave in the non-branching part of the channel approaches its limit value $\exp(-l\eta\varkappa b)$ obtained in /9/ as $\varkappa a$ increases.

A $\varkappa a$ increases when $\varkappa(a+b) = \pi n$ and $\varkappa a = \pi n$ $(n = 1, 2, \ldots)$ progressive waves are formed in the regions $x > 0$, $-b < y < a$ and $x < 0$, $0 < y < a$, respectively. The value of the dimensionless width at which a new propagating wave is generated is called the threshold value. On the graphs of the amplitude the threshold values of the width correspond to the characteristic kinks, which are due to the redistribution of energy between propagating waves close to the threshold of generation of a new progressive wave. The rearrangement of wave motions when a new propagating wave starts is well-known in optics /12/, electrodynamics /13/ and in nuclear physics, where it is called the threshold effect. In the theory of long surface waves this effect was first indicated in /10/. The threshold nature of the generation of progressive waves in channels was considered in /11, 15/.

The results of the present investigation can be used in geophysical calculations of tidal wave motion, as was done in /10/.

The author thanks V.A. Beliakov for discussing the results.

REFERENCES

1. SRETENSKII L.N., Theory of Wave Motions of Fluid. Moscow-Leningrad, Glav. Red. Obshchetekhn. Lit i Nomogr. ONTI, 1936.
2. PLIS V.I., On "conditions on the edge" in the linear theory of long surface waves. PMM, Vol.43, No.3, 1979.
3. NOBLE B., Methods based on the Wiener—Hopf Technique for the Solution of Partial Differential Equations. Pergamon Press, International Series of Maths, Vol.7, 1959.
4. MITTRA R. and LI S., Analytical Methods of the Theory of Waveguides. Moscow, MIR, 1974.
5. RIEKSTINYSH E.Ia., Asymptotic Expansion of Integrals. Vol.1, Riga, Zinatne, 1974.
6. GABOV S.A., Application of L.N. Sretenskii's method to a problem of the theory of waves in a channel. Zh. Vychicl. Matem. i Matem. Fiz. Vol.15, No.1, 1975.
7. KAPOULITSAS G.M., Scattering of long waves in a rotating branched channel. Internat. J. Theor. Phys. Vol.19, No.10, 1980.
8. KAPOULITSAS G.M., Diffraction of Kelvin waves from a rotating channel with an infinite and semi-infinite barrier. J. Phys., Ser. A, Vol.12, No.5, 1979.
9. GABOV S.A., RUBAN P. I., and SEKERZH-ZEN'KOVICH S.Ia., Diffraction of Kelvin waves on a semi-infinite wall in a semi-bounded tank. Zh. Vychicl. Matem. i Matem. Fiz., Vol.15, No.6,1975.
10. PACKHAM B.A., Reflection of Kelvin waves at the open end of a rotating semi-infite channel. J. Fluid. Mech., Vol.39, Pt.2, 1969.
11. PLIS A.I. and PLIS V.I., Diffraction of Kelvin waves at the open end of a plane-parallel channel. PMM Vol.44, No.1, 1980.
12. WOOD R.W., Diffraction gratings with controlled groove form and abnormal distribution of intensity. Philos. Mag. Ser. 6, Vol.23, No.133, 1912.
13. BOLOTOVSKII B.M. and LEBEDEV A.N., On threshold phenomena in classical electrodynamics. Zh. Experim. i Teor. Fiziki, Vol.53, No.4, 1976.
14. BAZ' A.I., ZEL'DOVICH Ia.B. and PERELOMOV A.M., Scattering, Reaction and Disintegration in Relativistic Quantum Mechanics. Moscow, NAUKA, 1966.
15. PLIS V.I., Propagation of Kelvin waves from a channel into a semibounded tank. PMM, Vol. 45, No.6, 1981.

Translated by J.J.D.